STABILITY OF PLANE-PARALLEL CONVECTIVE FLUID FLOW IN A HORIZONTAL LAYER RELATIVE TO SPATIAL PERTURBATIONS

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The stability of stationary plane-parallel convective flow between horizontal planes along which a constant temperature gradient is given, is investigated relative to spatial perturbations. It is shown that the flow crisis is caused by spiral perturbations in a broad range of Prandtl number values ($P > 0.24$). Spiral perturbations are developed in unstably stratified fluid layers adjoining the upper and lower layer boundaries, and are of Rayleigh nature.

1. The stability of a stationary plane-parallel convective flow between horizontal planes along which a constant temperature gradient is given is considered in [1] and the stability boundary relative to plane-normal perturbations is determined. The stability investigation of this flow is continued in this paper and spatial perturbations are examined.

A plane-parallel stationary flow originates in a horizontal layer bounded by solid planes along which the temperature varies linearly (Fig. 1). The dimensionless velocity and temperature (the measurement units are indicated in [1]) are distributed over the section as follows:

$$v_0 = {}^1/_6\,(x^3 - x), \qquad T_0 = z + GP\tau_0$$
$$\tau_0 = {}^1/_{360}\,(3x^5 - 10x^3 + 7x). \qquad (1.1)$$

Let us consider normal spatial perturbations of this flow which are dependent on the time and horizontal coordinates according to the law

$$(\mathbf{v}, T, p) \sim \exp\left[-\lambda t + i(k_y y + k_z z)\right]. \qquad (1.2)$$

The spectral problem for the amplitude follows from the linearized perturbation equations (the system (2.1)-(2.3) in [1]):

$$-\lambda v_x + ik_z G v_0 v_x = -p' + (v_x'' - k^2 v_x) - \theta$$
$$-\lambda v_y + ik_z G v_0 v_y = -ik_y p + (v_y'' - k^2 v_y)$$
$$-\lambda v_z + ik_z G v_0 v_z + G v_0' v_x = -ik_z p - (v_z'' - k^2 v_z)$$
$$-\lambda\theta - ik_z G v_0 \theta + G^2 P \tau_0' v_x + G v_z = P^{-1}(\theta'' - k^2\theta) \qquad (1.3)$$
$$v_x' + ik_y v_y + ik_z v_z = 0 \qquad (k^2 = k_y^2 + k_z^2)$$
$$x = \pm 1: \qquad v_x = v_y = v_z = \theta = 0$$

Here v_x, v_y, v_z, θ, and p are dependent on the transverse coordinate x of the perturbation amplitude, λ is the complex decrement, k_y, k_z are the wave numbers along the corresponding directions, and G and P are the Grashof and Prandtl numbers.

The plane perturbations case ($k_y = 0$, $k_z \neq 0$, $v_y = 0$) is considered in [1]. In contrast to the problem about spatial perturbations in a plane-parallel flow between parallel planes heated to different temperatures [2], the spectral problem (1.3) does not reduce to the corresponding problem for plane perturbations. Hence, the question of the behavior of spatial perturbations requires special consideration. By analogy with the results in [2], it can be expected that the most "dangerous" among the spatial perturbations are the spiral perturbations ($k_z = 0$, $k_y \neq 0$) in the form of shafts whose axes are parallel to the main flow velocity.

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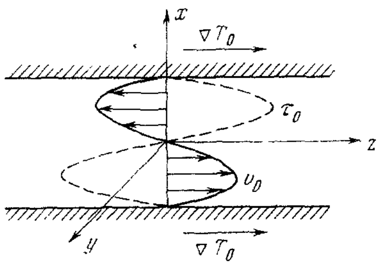

Fig. 1

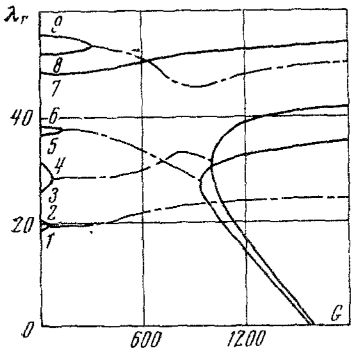

Fig. 2

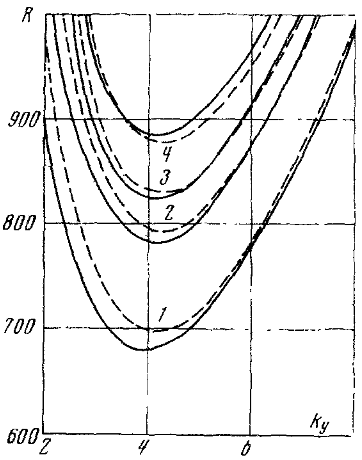

Fig. 3

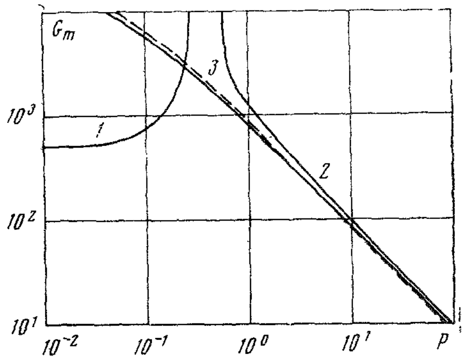

Fig. 4

Setting $k_z = 0$ and eliminating the amplitudes of v_y and p, we obtain the spectral problem for the spiral perturbations:

$$\begin{aligned} &\Delta^2 v_x - k_y^2\theta = -\lambda\Delta v_x \qquad (\Delta f \equiv f'' - k^2 f) \\ &\Delta v_z - G v_0' v_x = -\lambda v_z \\ &\Delta\theta - G^2P^2\tau_0' v_x - GPv_z = -\lambda P\theta \\ &x = \pm 1: \quad v_x = v_x' = v_z = \theta = 0\ . \end{aligned} \tag{1.4}$$

Because of the oddness of the velocity v_0 and temperature τ_0 profiles, the boundary-value problem (1.4) admits of solutions of two kinds, even and odd relative to the middle of the layer.

2. The Galerkin method is used to solve the problem (1.4). The approximations of the amplitudes of v_x, v_z and θ are

$$v_x = \sum_{m=0}^{M-1} a_m\varphi_m, \quad \theta = \sum_{n=0}^{N-1} b_n\theta_n, \quad v_z = \sum_{l=0}^{L-1} c_l\sigma_l\ . \tag{2.1}$$

The basis functions φ_m, θ_n, σ_l (they have the meaning of perturbation amplitudes in a fixed fluid layer) are determined by solutions of the boundary-value problems

$$\begin{aligned} &\Delta^2\varphi_m = -\mu_m\,\Delta\varphi_m, \qquad \varphi_m(\pm 1) = \varphi_m'(\pm 1) = 0 \\ &\Delta\theta_n = -\nu_n P\theta_n, \qquad \theta_n(\pm 1) = 0 \\ &\Delta\sigma_l = -\chi_l\sigma_l, \qquad \sigma_l(\pm 1) = 0\ . \end{aligned} \tag{2.2}$$

The approximations (2.1) contain from four to 20 basis functions in expansions of v_x, θ, v_z.

The fundamental results of the computations are represented in Figs. 2-4. Shown in Fig. 2 are the real parts of the perturbation decrements λ_r as a function of the Grashof number G for fixed parameters $P = 0.5$, $k_y = 4$. The solid lines show the real branches of the spectrum; the dash-dot lines denote the common real part of the pairs of complex-conjugate decrements. All the levels are real for $G = 0$ and alternate as follows in order of increasing decrements: χ_0, μ_0, χ_1, μ_1, ν_0, χ_2, μ_2, ν_1, χ_3. As is seen from the spectrum, there are two critical points at which an instability of monotonic form originates (the real decrement becomes zero). The lower (even) instability level is generated by a mixture of even branches of ν_0 and χ_2. The higher (odd) level is generated by a mixture of odd branches of χ_1 and μ_1. The critical Grashof numbers are close together. The monotonic nature of the instability, the presence of two critical points, and their closeness are conserved even for other values of the parameters P and k_y.

Shown in Fig. 3 are the neutral curves on the plane of the critical Rayleigh number R — wave number k_y (R = GP). The solid lines map the neutral curves for the even instability level; the dashed lines refer to an odd level. The numbers 1-4 of the pairs of neutral curves correspond to values of the Prandtl numbers 0.2, 0.5, 1, and 10. It is seen that the critical wave number is close to $k_m \approx 4$ in a broad range of values of P.

The combined stability diagram is presented in Fig. 4. The dependence of the minimal critical Grashof number G_m on the Prandtl number P is shown for different instability modes. Curve 1 yields the stability boundary relative to plane monotonic perturbations of hydrodynamic type; curve 2 determines the stability boundary relative to plane traveling perturbations of Rayleigh type (curves 1 and 2 are found in [1]). The pair of curves 3 refers to the spatial spiral perturbations discussed in this paper. The solid and dashed lines, respectively, map the stability boundary relative to even and odd perturbations. The spiral perturbations are the most dangerous among all the kinds of perturbations considered in the broad range of Prandtl numbers $P > 0.24$ and result in a crisis of the plane-parallel flow.

The minimal critical numbers G_m for the even and odd spiral perturbations differ slightly. Even-type perturbations are more dangerous for $P < 2.7$; while the crisis goes over to the odd perturbation for $P > 2.7$. The stability boundaries in this range of Prandtl numbers are so close together that they practically coincide in the scale in Fig. 4.

The critical numbers G_m diminish monotonically as P grows, and for large P the following asymptotic is valid:

$$G_m = a / P. \qquad (2.3)$$

where the coefficient a is 886 and 879, respectively, for even and odd levels.

Numerical investigations of the eigenfunctions show that a spiral instability originates because of the development of perturbations in unstably stratified fluid layers adjoining the upper and lower planes (Fig. 1). Computations of the streamlines in the x-y plane perpendicular to the main flow direction show that two fundamental vortices localized in the upper and lower halves of the channel section are formed at a half-wavelength π/k_y in the y direction. In the case of an odd perturbation, two vortices, one above the other, have opposite circulation directions. In the even perturbation case, the main vortices have the same circulation direction and a weak buffer vortex of opposite circulation is formed between them.

The spiral instability is due to the equilibrium crisis of the fluid heated from below, just as the instability relative to plane traveling waves (curve 2 in Fig. 4). The unstable temperature stratification is produced by plane-parallel flow. This flow exerts influence on the condition for the origination of a Rayleigh-type instability.

The Rayleigh nature of the spiral instability is verified by the fact that the Rayleigh number [formula (2.3)] is the governing parameter for large P. The boundary-value problem (1.4) can be simplified for large P. It follows from the first two equations of the system (1.4) that the amplitudes are on the order of ($\lambda = 0$) on the stability boundary $\theta \sim v_x$, $v_z \sim G v_x \sim P^{-1} v_x$. The first and third equations then become

$$\Delta^2 v_x = k_y^2 \theta. \qquad \Delta\theta = R^2 \tau_0' v_x. \qquad (2.4)$$

Together with the appropriate boundary conditions, these equations govern the neutral perturbations in a fixed fluid with a vertical temperature gradient τ_0'. In contrast to the spiral perturbations, the plane traveling perturbations considered in [1] are also a Rayleigh instability on which the main stream acts in a stabilizing manner.

LITERATURE CITED

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